EFFECT OF SCREW DISLOCATION ON THE PROPAGATION OF WAVES IN AN ELASTIC CYLINDER

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We shall investigate the characteristics of wave propagation in an infinite hollow circular cylinder with initial stresses produced by the presence of a screw dislocation in the cylinder. The initial stressed state, in which the lateral surfaces of the cylinder are free of a load, is determined from the exact solution of the problem of a screw dislocation with finite deformations. This solution is found for an arbitrary, isotropic, nonlinear-elastic, incompressible (including radially nonuniform) material. The equations of small oscillations around the equilibrium state described are formulated. These equations have solutions in the form of dispersive waves propagating along the axis of the cylinder. The construction of the dispersion relation reduces to a solution of a homogeneous boundary value problem for a system of ordinary differential equations; this is done by a numerical method. In the numerical examples, we use a model of a material with an elastic potential in Mooney's form.

Initial State. The isochoric shear and tensile deformations of a cylindrical panel are described by the relations [1]

$$\lambda\left(r^2-r_0^2\right)=\rho^2-\rho_0^2,\ \theta=\varphi,\ z=\alpha\varphi+\lambda\zeta, \tag{1}$$

where ρ, φ, and ζ are the cylindrical coordinates of the undeformed state; r, θ, z are the cylindrical coordinates of points on the body after deformations; λ, α are constants; ρ_0, r_0 are the outer radius of the cylinder before and after deformation, respectively. A deformation of the form (1) can be realized also for a cylindrical shell which is closed in the angular direction. For this, the shell must be cut by the half-plane $\theta = 0$, the edges of the cut must be displaced relative to one another along the axis of the cylinder by an amount $2\pi\alpha$, and then the surface of the cut must be fixed. The indicated deformation creates in the cylinder a screw dislocation [2] with the Burgers vector oriented along the axis of the cylinder with length $2\pi\alpha$. We note that, in contrast to the linear theory of elasticity [2], here the magnitude of Burgers vector is not assumed to be small, i.e., arbitrarily large deformations are studied.

The determining relation of the isotropic incompressible elastic material has the form [1, 3]

$$\mathbf{T}=\varkappa_1(I_1,\ I_2,\ \rho)\mathbf{F}-\varkappa_2(I_1,\ I_2,\ \rho)\mathbf{F}^{-1}-p\mathbf{E},\quad I_1=\operatorname{tr}\mathbf{F},\ I_2=\operatorname{tr}(\mathbf{F}^{-1}), \tag{2}$$

where **T** is the Cauchy stress tensor; **F** is Finger's deformation measure; **E** is a unit tensor; I_1 and I_2 are the first and second invariants of the tensor **F** (the third invariant equals one by virtue of the condition of incompressibility); p is the pressure in the incompressible body, not determined by the deformation; $\varkappa_1$, and $\varkappa_2$ are some functions of the invariants defining the specific material. The explicit form of these functions of ρ permits taking into account the possible radial nonuniformity of the material.

The expressions for the measures of deformation, corresponding to (1), have the form

$$\mathbf{F}=\frac{\rho^2}{\lambda^2r^2}\mathbf{f}_1\mathbf{f}_1+\frac{r^2}{\rho^2}\mathbf{f}_2\mathbf{f}_2+\frac{\alpha r}{\rho^2}(\mathbf{f}_2\mathbf{f}_3+\mathbf{f}_3\mathbf{f}_2)+\left(\lambda^2+\frac{\alpha^2}{\rho^2}\right)\mathbf{f}_3\mathbf{f}_3, \tag{3}$$

$$\mathbf{F}^{-1}=\frac{\lambda^2r^2}{\rho^2}\mathbf{f}_1\mathbf{f}_1+\frac{\rho^2}{r^2\lambda^2}\left(\lambda^2+\frac{\alpha^2}{\rho^2}\right)\mathbf{f}_2\mathbf{f}_2-\frac{\alpha}{\lambda r^2}(\mathbf{f}_2\mathbf{f}_3+\mathbf{f}_3\mathbf{f}_2)+\frac{1}{\lambda^2}\mathbf{f}_3\mathbf{f}_3.$$

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Here $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$ is the orthonormal vector basis of the cylindrical coordinates in the deformed configuration; in addition, the first index corresponds to the radial coordinate, the second corresponds to the angular coordinate, and the third corresponds to the axial coordinate.

From relations (2) and (3) and the equations of equilibrium it follows [1] that the tangential stresses τ_{12} and τ_{13} vanish, and the remaining components of the tensor T in the basis $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$ depend only on the coordinate r.

If the exterior surface $\rho = \rho_0$ of the cylinder is not loaded, then the expressions for the stresses and the function p satisfy the conditions of equilibrium, and have the form

$$\sigma_1 = \int_r^{r_0} \left[\varkappa_1(r) \left(\frac{\rho^2}{\lambda^2 r^2} - \frac{r^2}{\rho^2} \right) - \varkappa_2(r) \left(\frac{\lambda^2 r^2}{\rho^2} - \frac{\rho^2}{r^2} - \frac{\alpha^2}{\lambda^2 r^2} \right) \right] \frac{dr}{r}, \tag{4}$$

$$\sigma_2 = \varkappa_1(r) \left(\frac{r^2}{\rho^2} - \frac{\rho^2}{\lambda^2 r^2} \right) - \varkappa_2(r) \left(\frac{\rho^2}{r^2} + \frac{\alpha^2}{\lambda^2 r^2} - \frac{\lambda^2 r^2}{\rho^2} \right) + \sigma_1,$$

$$\sigma_3 = \varkappa_1(r) \left(\lambda^2 + \frac{\alpha^2}{\rho^2} - \frac{\rho^2}{\lambda^2 r^2} \right) - \varkappa_2(r) \left(\frac{1}{\lambda^2} - \frac{\lambda^2 r^2}{\rho^2} \right) + \sigma_1,$$

$$\tau_{23} = \varkappa_1(r) \frac{\alpha r}{\rho^2} + \varkappa_2(r) \frac{\alpha}{\lambda^2 r}, \quad p = \varkappa_1(r) \frac{\rho^2}{\lambda^2 r^2} - \varkappa_2(r) \frac{\lambda^2 r^2}{\rho^2} - \sigma_1.$$

Expressions for the functions $\varkappa_1(r)$, $\varkappa_2(r)$ can be obtained from relations (1)-(3). The constant r_0 is determined from the boundary condition $\sigma_1(r_1) = 0$, where r_1 is the inner radius of the cylindrical shell in the deformed state.

The stresses acting on any transverse cross section of the cylinder are statically equivalent to a longitudinal force P and a torque M, which are functions of the parameters λ, α:

$$P(\alpha, \lambda) = 2\pi \int_{r_1}^{r_0} \sigma_3(r)\, r dr, \quad M(\alpha, \lambda) = 2\pi \int_{r_1}^{r_0} \tau_{23}(r) r^2\, dr. \tag{5}$$

The coefficient of axial stretching λ is then determined from the condition $P = 0$. A numerical analysis for Mooney's material showed that any value of the parameter α corresponds to a value of $\lambda < 1$, i.e., with axial displacement under conditions such that there is no longitudinal force, the cylindrical shell is compressed in the axial direction.

Linearized Equations of Motion. Let us assume that a small motion, defined by the displacement vector $\eta\mathbf{w}$, where η is a small parameter, is superposed on the initial deformed state. Then the linearized equations of motion are described by the equations [1, 3]

$$\nabla \cdot \boldsymbol{\Theta} = m \frac{\partial^2 \mathbf{w}}{\partial t^2}, \quad \boldsymbol{\Theta} = \mathbf{T}^{\cdot} - (\nabla \mathbf{w})^T \cdot \mathbf{T}, \quad \nabla \cdot \mathbf{w} = 0, \tag{6}$$

$$\mathbf{T}^{\cdot} = \frac{d}{d\eta} \mathbf{T}(\mathbf{R} + \eta \mathbf{w})\big|_{\eta=0}, \quad \nabla = \mathbf{f}_1 \frac{\partial}{\partial r} + \mathbf{f}_2 \frac{\partial}{r\partial\theta} + \mathbf{f}_3 \frac{\partial}{\partial z},$$

where t is the time, m is the density of the material, $\mathbf{R}$ is the radius vector of the starting stress state, and ∇ is the nabla operator in the metric of the prestressed body.

Since the lateral surfaces of the shell are assumed to be free of loads, the boundary conditions for Eqs. (6) have the form

$$\mathbf{f}_1 \cdot \boldsymbol{\Theta} = 0 \ \text{at} \ r = r_1, r_0. \tag{7}$$

The equations (6) have solutions of the form

$$\begin{Bmatrix} u \\ v \\ w \\ q \end{Bmatrix} = e^{i(\theta n + z\nu/r_0 - t\omega)} \begin{Bmatrix} U(r) \\ V(r) \\ W(r) \\ Q(r) \end{Bmatrix}. \tag{8}$$

Here ν is a dimensionless wave number, n is an integer, ω is the frequency of oscillation, $\mathbf{w} = uf_1 + vf_2 + wf_3$, $q = -p^{\cdot}$.

Substitution of the solution (8) into Eq. (6) and the boundary conditions (7) gives a homogeneous boundary value problem for the functions U, V, W, and Q for an isotropic incompressible material of general form. For a homogeneous cylinder consisting of Mooney's material ($\varkappa_1 = 2C_1$, $\varkappa_2 = 2C_2$; C_1, C_2 = const), this boundary-value problem is written as follows:

$$\begin{gathered}
U''\left(\beta_1+3\varkappa\beta_1^{-1}\right)+U'\left[\beta_1'+3\varkappa\left(\beta_1^{-1}\right)'+p/\varkappa_1+R^{-1}\left(\beta_1+3\beta_1^{-1}\varkappa\right)\right]+ \\
+U\left[-\left(1+n^2\right)\lambda^{-2}\beta_1^{-1}R^{-2}+2in\alpha\beta_2\nu R^{-1}+\nu^2\left(\beta_3+\varkappa\beta_1^{-1}\right)-\varkappa R^{-2}\left(3\beta_1\beta_3+n^2\beta_1^{-1}\right)+\right. \\
\left.+mr_0^2\omega^2/\varkappa_1\right]+V'\left[\left(\beta_1^{-1}+\beta_1\beta_3\right)in\varkappa R^{-1}-\varkappa\nu\mu x^{-1}\lambda^{-2}R^{-1}\right]+V\left[-2in\lambda^{-2}\beta_1^{-1}R^{-2}-\right. \\
\left.-2\nu\alpha\beta_2R^{-1}+\varkappa R^{-2}\lambda^{-2}x^{-1}\nu\mu-in\varkappa R^{-2}\left(3\beta_1\beta_3+\beta_1^{-1}\right)\right]+W'\left[-\varkappa R^{-2}\lambda^{-2}x^{-1}in\mu+\right. \\
\left.+\varkappa\nu\left(\lambda^{-2}+\beta_1^{-1}\right)\right]+2\varkappa in\mu\lambda^{-2}R^{-3}x^{-1}W+Qr_0/\varkappa_1=0, \\
V''\left(\beta_1+\varkappa\beta_1\beta_3\right)+V'\left[\beta_1'+\varkappa\left(\beta_1\beta_3\right)'+\beta_1R^{-1}\left(1+\varkappa\beta_3\right)\right]+V\left[2in\alpha\beta_2R^{-1}-\right. \\
-\lambda^{-2}\beta_1^{-1}R^{-2}\left(1+n^2\right)+\beta_3\nu^2-p'R^{-1}/\varkappa_1-\varkappa R^{-1}\left(\beta_1^{-1}+\beta_1\beta_3\right)'-3n^2\varkappa\beta_1\beta_3R^{-2}- \\
\left.-\varkappa\beta_1^{-1}R^{-2}-2in\varkappa\mu\nu\lambda^{-2}R^{-2}x^{-1}+\varkappa\beta_1\beta_3\nu^2+mr_0^2\omega^2/\varkappa_1\right]+U'\varkappa R^{-1}\left[in\left(\beta_1^{-1}+\beta_1\beta_3\right)-\right. \\
\left.-\mu\nu\lambda^{-2}x^{-1}\right]+U\left[inR^{-1}p'/\varkappa_1+2in\lambda^{-2}R^{-2}\beta_1^{-1}+2\alpha\beta_2\nu R^{-1}-\varkappa\nu\mu\lambda^{-2}R^{-2}x^{-1}+\right. \\
\left.+in\varkappa R^{-2}\left(3\beta_1\beta_3+\beta_1^{-1}\right)\right]-\varkappa\mu\lambda^{-2}R^{-1}x^{-1}\left(W''+R^{-1}W'\right)+W\left[2n^2\varkappa\mu x^{-1}\lambda^{-2}R^{-3}-\right. \\
\left.-2\mu\nu^2\varkappa x^{-1}\lambda^{-2}R^{-1}+in\lambda^{-2}\nu\varkappa R^{-1}+\beta_1\beta_3\varkappa in\nu R^{-1}\right]+inR^{-1}r_0Q/\varkappa_1=0, \\
W''\left(\beta_1+\lambda^{-2}\varkappa\right)+W'\left[\beta_2'+\beta_1R^{-1}+\lambda^{-2}\varkappa R^{-1}\right]+W\left[\beta_3\nu^2+3\lambda^{-1}\nu^2\varkappa-\lambda^{-2}\beta_1^{-1}n^2R^{-2}+\right. \\
\left.+2\alpha\beta_2in\nu R^{-1}-\varkappa n^2\lambda^{-2}R^{-2}-2in\varkappa\mu\nu x^{-1}\lambda^{-2}R^{-2}+mr_0^2\omega^2/\varkappa_1\right]+U'\left[\varkappa\beta_1^{-1}\nu+\right. \\
\left.+\varkappa\nu\lambda^{-2}-in\varkappa\mu x^{-1}\lambda^{-2}R^{-2}\right]+U\left[\nu p'/\varkappa_1+\varkappa\nu\left(\beta_1^{-1}\right)'-in\varkappa\mu\lambda^{-2}R^{-3}x^{-1}+\right. \\
\left.+\varkappa\nu\beta_1^{-1}R^{-1}+\varkappa\nu\lambda^{-2}R^{-1}\right]-\varkappa\mu\lambda^{-2}R^{-1}x^{-1}\left(V''-R^{-1}V'\right)+V\left[\mu\varkappa x^{-1}\lambda^{-2}R^{-3}\left(2n^2-\right.\right. \\
\left.\left.-1\right)+\varkappa\beta_1\beta_3in\nu R^{-1}+in\nu\varkappa\lambda^{-2}R^{-1}-2\mu\nu^2\varkappa x^{-1}\lambda^{-2}R^{-1}\right]+Q\nu r_0/\varkappa_1=0;
\end{gathered} \tag{9}$$

$$U' + R^{-1}U + inR^{-1}V + \nu W = 0. \tag{10}$$

The following notation is used in the formulas (9): $\beta_1 = \rho^2/(\lambda r)^2$, $\beta_2 = \alpha r/\rho^2$, $\beta_3 = \lambda^2 + \alpha^2/\rho^2$, $R = r/r_0$, $\varkappa = \varkappa_2/\varkappa_1$, $\mu = \alpha/\rho_0$, $x = r_0/\rho_0$. The prime indicates differentiation with respect to the dimensionless variable R. Equation (10) is the condition of incompressibility, written in a cylindrical coordinate system.

The boundary conditions on the lateral surfaces are as follows:

$$\begin{gathered}
\left(\beta_1+3\varkappa\beta_1^{-1}+p/\varkappa_1\right)U'+Qr_0/\varkappa_1=0, \\
\left(\beta_1+\varkappa\beta_1\beta_3\right)V'+V\left[-pR^{-1}/\varkappa_1-\varkappa R^{-1}\left(\beta_1^{-1}+\beta_1\beta_3\right)\right]-W'\mu\varkappa x^{-1}\lambda^{-2}R^{-1}+ \\
+UR^{-1}\left[in\varkappa\left(\beta_1^{-1}+\beta_1\beta_3\right)-\mu\nu\varkappa\lambda^{-2}x^{-1}+inp/\varkappa_1\right]=0, \\
\left(\beta_1+\lambda^{-2}\varkappa\right)W'-\mu\varkappa\lambda^{-2}R^{-1}x^{-1}V'+\mu\varkappa\lambda^{-2}R^{-2}x^{-1}V+U\left[\varkappa\nu\left(\beta_1^{-1}+\lambda^{-2}\right)-in\mu\varkappa\lambda^{-2}R^{-2}x^{-1}+p\nu/\varkappa_1\right]=0, \\
R=1,\ r_1/r_0.
\end{gathered} \tag{11}$$

The condition for the absence of compressive forces, by virtue of (4) and (5), leads to the equality

$$\begin{gathered}
\left(1-\chi^2\right)\left(2\lambda-\lambda^{-2}-\lambda^{-3}x^2\right)-2\mu^2\lambda^{-1}\ln\chi^2+\chi^2\lambda^{-2}\ln\frac{\lambda x^2+\chi^2-1}{\lambda x^2}- \\
-\chi^2\lambda^{-2}\ln\chi^2+\varkappa\left[\left(1-\chi^2\right)\left(3-2\lambda^{-3}-\lambda^{-1}x^{-2}-\mu^2\lambda^{-2}x^{-2}\right)+\left(2-2\lambda x^2-\chi^2\right)\ln\chi^2+\left(2\lambda x^2-2+\chi^2-\mu^2\lambda^{-2}\right)\ln\frac{\lambda x^2+\chi^2-1}{\lambda x^2}\right]=0,
\end{gathered} \tag{12}$$

where $\chi = \rho_1/\rho_0$.

Nontrivial solutions of the boundary-value problem (9)-(11) exist only for a definite dispersion relation: the dependence of the phase velocity of the wave $\omega r_0/\nu$ on the frequency ω.

The case n = 1 corresponds to waves of bending–torsional oscillations of the cylinder.

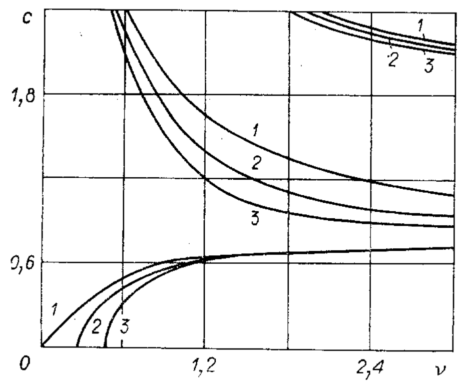

Fig. 1

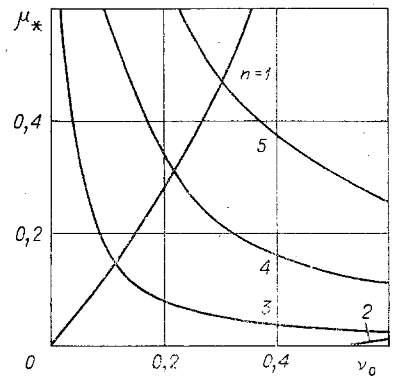

Fig. 2

The boundary-value problem (9)-(11) with condition (12) is numerically solved by a method which in many ways is analogous to [4].

In order to proceed to the investigation of the stability of a cylindrical shell subjected to a screw dislocation, it is sufficient to set $\omega = 0$ in Eqs. (8).

In this case, the value n = 1 corresponds to a rod-shaped bulging of the cylinder. As in [4], it can be shown that solutions of the form (8) with n = 1 in the sections of the cylinder $z = 0$, $z = l = \pi r_0/\nu$ satisfy the conditions of hinged support along one axis of the transverse section and conditions of slipping support along the other axis, orthogonal to the first one.

Numerical Results. We shall restrict ourselves to a presentation of the results for a cylindrical shell consisting of a non-Hookian material. For this, it is sufficient to set $\varkappa_2 = 0$ in the relations (9)-(11). The results of the calculations for n = 1 and for a ratio of the inner radius to the outer radius equal to 0.8 in the undeformed state are presented in Fig. 1. The first three dispersion curves are constructed. The wave number ν is plotted along the abscissa axis, and the dimensionless value of the phase velocity of propagation of bending—torsional waves with $c = (m/\varkappa_1)^{1/2}\omega r_0/\nu$ is plotted along the ordinate axis. The numbers 1-3 denote the curves corresponding to a definite value of the parameter μ, which characterizes the magnitude of Burgers vector (respectively, $\mu = 0, 0.3, 0.5$). It turns out that the higher the initial deformation, the lower the phase velocity of the wave. The points of intersection of the curves with the abscissa axis are points of elastic instability (bifurcation of the equilibrium) of the cylinder, caused by the presence of the screw dislocation.

The results for the problem of stability are presented in greater detail in Fig. 2. The dependences of the critical magnitude of the deformation μ_* on the parameter ν_0, related to the geometric parameters by the relation $\nu_0 = \pi\lambda\rho_0/l$, where $2l$ is the wavelength of the bulging deformation wave, are constructed for $\chi = 0.9$ for n = 1, 2, 3, 4, and 5, which characterizes the form of the loss of stability. The rodlike form of the bulging (n = 1) and of the shape determined by the value n = 2 is characterized by the increase in the parameter μ_* with increasing ν_0. The opposite holds for forms for which n > 2.

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